Music 170: Definitions and formulas for week 1

Definitions

Velocity is distance travelled per unit time:

$$v = \Delta x / \Delta t$$

in units of m/sec and

acceleration is velocity change per unit time:

$$a = \Delta v / \Delta t$$

in units of m/sec^2.

Force is acceleration times mass (in Newtons, i.e., Kg. meter per second squared)

Pressure is force per unit area (in Newtons per square meter)

$$P = F/A$$

Density is mass per unit volume (Kg per cubuic meter)

$$\rho = m/V$$

Formulas (with some derivations)

Sinusoidal motion. Here, "x" could mean position or any other quantity. We say that x behaves as a sinusoid when:

$$x(t) = A \cdot \cos\left(\omega t + \phi\right).$$

Here A is the *amplitude*, ω the *angular frequency*, and ϕ the *initial phase*. The angular frequency is in radians per second, and the initial phase in radians. The **cycle frequency** is

$$f = \frac{\omega}{2\pi}.$$

We can re-write the expression for sinusoidal motion as:

$$x(t) = A \cdot \cos\left(2\pi f t + \phi\right).$$

The frequency f is in units of cycles per second.

Velocity and acceleration of sinusoids

If $x(t)$ is as above, the velocity is:

$$v(t) = \omega A \cos\Big(\omega t + (\phi - \frac{\pi}{2})\Big)$$

and the acceleration is:

$$a(t) = \omega^2 A \cos(\omega t + (\phi - \pi))$$

Vibration frequency of the simple mass/spring system

Suppose a mass is connected to a spring, the other end of which is fixed. There is one mode, whose frequency we'll now derive in two ways. Force f is related to displacement $x = x(t)$ by Hooke's law:

$$F = -Kx,$$

and to acceleration by Newton's second law:

$$F = ma.$$

If we assume that x is a sinusoid, this gives:

$$-KA \cdot \cos(\omega t + \phi) = m\omega^2 A \cos(\omega t + (\phi - \pi))$$

and so:

$$\omega = \sqrt{\frac{K}{m}}.$$

The vibration frequency, in cycles per second, is:

$$f = \frac{1}{2\pi}\sqrt{K/m}$$

Second derivation

It's also possible to find f using conservation of energy. The potential energy is:

$$E_p = \frac{K}{2}x^2$$

and kinetic energy is:

$$E_k = \frac{m}{2}v^2$$

At the moment when all the energy is kinetic, the total is

$$E_k = \frac{m}{2}\omega^2 a^2$$

and when it is all potential it equals:

$$E_p = \frac{K}{2}a^2.$$

Solving for ω then gives the same formula for f as before. This analysis works better than the earlier one when the situation gets more complicated, e.g., when we show how the springiness of air makes sound propagate.

Resonance frequency for Helmholz resonator

Here is a derivation of the resonance frequency of a Helmholz resonator, assuming an ideal gas. Suppose the gas has density ρ and pressure P. The resonator itself has a chamber of volume V and an opening of area a. Its neck has a length l.

Mass is neck volume times air density

$$m = la\rho,$$

Force is change in pressure times area. If the air moves down d units:

$$\Delta F = \left[\frac{V + ad}{V} P - P\right] a$$

$$= \frac{a^2 P d}{V}$$

and the spring constant K is Δf divided by d:

$$K = \frac{a^2 P}{V}$$

Frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{K}{m}} = \frac{1}{2\pi}\sqrt{\frac{P}{\rho}}\sqrt{\frac{a}{Vl}}$$

$$= \frac{v}{2\pi}\sqrt{\frac{a}{Vl}}$$

where v is the velocity of sound in the ideal gas. We won't derive that here, but it turns out that

$$v = \sqrt{P/\rho}$$

In real air (as opposed to an ideal gas), the spring constant is about 1.4 times the ideal gas value (this is the factor of 1.4 that appears in Rossing.) This factor is also reflected in the "real" velocity of sound so that the formula for frequency in terms of v still works.

Here is a sample calculation: a bottle has a neck 1.7 cm across (so the area is $a = 2.27\text{cm}^2$ and a length of $l = 6cm$, and and holds a volume $v = 150cc$. Then we get:

$$\sqrt{\frac{a}{Vl}} = 0.05/\text{cm} = 5/\text{meter}$$

so, if the speed of sound is 343 M/sec, we get:

$$f = 343 \cdot 5/(2 * 3.14) = 273/\text{sec}$$